

UNCLASSIFIED

AD 402 212

*Reproduced
by the*

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA



UNCLASSIFIED

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

63-3-2

OFFICE OF NAVAL RESEARCH

Contract Nonr 562(25)

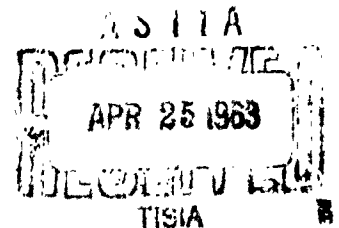
NR-064-431

Technical Report No. 20

**A GENERALIZATION OF THE BELTRAMI STRESS FUNCTIONS
IN CONTINUUM MECHANICS**

by

M. E. Gurtin



DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

PROVIDENCE, R. I.

April 1963

CATALOGED BY ASTIA
AS AD NO. 402212

A generalization of the Beltrami stress functions
in continuum mechanics¹

by

M. E. Gurtin
Brown University

1. Introduction.

In the absence of body-forces the stress equations of equilibrium for a continuous medium take the form

$$\tau_{km,m} = 0, \quad \tau_{km} = \tau_{mk} \quad (1.1)$$

when referred to rectangular cartesian coordinates.² Apparently the first stress-function solution of (1.1) was Airy's [1863] two-dimensional solution in terms of a single stress function. Three dimensional generalizations of Airy's stress function were obtained by Maxwell [1870] and Morera [1892], who established two alternative solutions of the stress equations of equilibrium, each involving a triplet of stress functions. Beltrami [1892] observed that the solutions due to Maxwell and Morera may be regarded as special cases of a solution to (1.1) which may be written as

¹ The results communicated in this paper were obtained in the course of an investigation conducted under Contract Nonr-562(25) of Brown University with the Office of Naval Research.

² Here as well as in the sequel we use the usual indicial notation. Thus, Latin subscripts have the range of the integers (1,2,3) and summation over repeated subscripts is implied; subscripts preceded by a comma indicate differentiation with respect to the corresponding cartesian coordinate. Further, ϵ_{ijk} and δ_{ij} respectively denote the alternating symbol and Kronecker's delta.

$$\tau_{km} = \varepsilon_{kip} \varepsilon_{mj} \varphi_{ij,pq}, \quad \varphi_{ij} = \varphi_{ji}. \quad (1.2)^3$$

Equation (1.2) degenerates into Maxwell's solution if one takes $\varphi_{ij} = 0$ ($i \neq j$); it reduces to Morera's solution if one sets $\varphi_{ii} = 0$ (no sum), and it yields Airy's solution if one assumes that all of the φ_{ij} , except φ_{33} , vanish.

Completeness proofs for Beltrami's solution were given by Morinaga and Nôno [1950], Ornstein [1954], Günther [1954], and Dorn and Schild [1956]. All of these proofs, however, are valid only when the boundary of the region under consideration consists of a single closed surface. This fact was noted by Rieder [1960], who showed that the representation (1.2) is incomplete. In fact, as shown by Rieder, (1.2) can at most represent solutions of (1.1) which are "totally self-equilibrated" in the sense that they correspond to zero resultant force and moment on every closed surface contained in the region. It is at once clear that solutions which are not totally self-equilibrated do indeed exist whenever the boundary of the region consists of more than a single closed surface.

³ Beltrami's solution was later independently arrived at by Gwyther [1912] and Finzi [1934] and has occasionally been attributed to them. Actually, the tensorial version (1.2) of Beltrami's solution seems to have made its first appearance in Finzi's [1934] paper. The result (1.2) was also rediscovered by Weber [1948], Morinaga and Nôno [1950], Schaefer [1953], and Ornstein [1954]. Related contributions, as well as additional references, may be found in publications by Kuzmin [1945], Krutkov [1949], Blokh [1950], Langhaar and Stippes [1954], Kröner [1954], Marguerre [1955], Truesdell [1959], and Sternberg [1960].

Section 2 of the present paper contains certain preliminary definitions and notational agreements. In Section 3 we include a concise version of Rieder's result and subsequently use a minor variant of Günther's argument to prove that every (sufficiently smooth) totally self-equilibrated solution of (1.1) admits the representation (1.2).

In Section 4 we first introduce the stress-function solution of (1.1) given by

$$\left. \begin{aligned} \tau_{km} &= \epsilon_{kip} \epsilon_{mjq} \varphi_{ij,pq} + \nabla^2 (\psi_{k,m} + \psi_{m,k}) - \psi_{j,jkm} \\ \varphi_{ij} &= \varphi_{ji} \quad , \quad \nabla^4 \psi_k = 0 \quad . \end{aligned} \right\} \quad (1.3)$$

We then prove that this solution is complete in the sense that every (not necessarily totally self-equilibrated) solution of (1.1) may be represented in the form (1.3). It follows as an immediate corollary of this result that every solution of (1.1) may be decomposed into the sum of a totally self-equilibrated solution and a biharmonic solution.

2. Preliminary definitions. Equilibrated stress fields.

Throughout the paper \mathcal{Q} will denote a bounded region of three-dimensional Euclidean space. The region \mathcal{Q} will be either open or closed. On the other hand R will always designate a bounded open region whose closure is \bar{R} . We call \bar{R} a regular region if its boundary consists of a finite number of non-intersecting closed regular surfaces, the latter term being used in the sense of Kellogg [1929] (p.112).

We write $f \in C^N(\mathcal{Q})$ if and only if f is a real-valued function defined and N times continuously differentiable on \mathcal{Q} .

We say that τ_{km} is an equilibrated stress field on R if and only if $\tau_{km} \in C^1(R)$ and

$$\tau_{km,m} = 0, \quad \tau_{km} = \tau_{mk} \quad \text{on } R. \quad (2.1)$$

We say that τ_{km} is an equilibrated stress field on \bar{R} if and only if τ_{km} is an equilibrated stress field on R and τ_{km} is continuous on \bar{R} .

Suppose τ_{km} is an equilibrated stress field on \mathcal{Q} and S a closed regular surface contained in \mathcal{Q} . Further let T_k be the surface traction of τ_{km} on S defined at every regular point of S through

$$T_k = \tau_{km} n_m, \quad (2.2)$$

n_m being the unit outward normal of S . Then we call the vector $\underline{L}(S)$ with components

$$L_k(S) = \int_S T_k dA \quad (2.3)$$

the resultant force of τ_{km} on S and the vector $\underline{M}(S)$ with components

$$M_k(S) = \int_S \varepsilon_{kij} x_i T_j dA \quad (2.4)$$

the resultant moment (about the origin) of τ_{km} on S .

An equilibrated stress field on \mathcal{Q} will be called totally self-equilibrated if and only if its resultant force and moment vanish on every closed regular surface contained in \mathcal{Q} .

The following elementary result further clarifies the difference between equilibrated and totally self-equilibrated stress fields.

2.1 Theorem. Let \bar{R} be a regular region whose boundary consists of the closed surfaces B_α ($\alpha=1,2,\dots,N$) and let τ_{km} be an equilibrated stress field on \bar{R} . Then τ_{km} is totally self-equilibrated if and only if its resultant force and moment vanish on each B_α .

Proof. If τ_{km} is totally self-equilibrated then, by definition,

$$\underline{L}(B_\alpha) = \underline{M}(B_\alpha) = 0 \quad (\alpha=1,2,\dots,N). \quad (2.5)$$

Conversely, (2.5), by virtue of (2.1), (2.3), (2.4), and the divergence theorem, implies that τ_{km} is totally self-equilibrated.

We now turn to a characterization of the type of region on which there exist equilibrated stress fields which are not totally self-equilibrated. Let S henceforth denote a closed regular surface. We say that \mathcal{Q} is periphRACTIC⁴ if and only if it contains an S enclosing at least one point that does not belong to \mathcal{Q} . Thus a periphRACTIC region has "holes". The spherical shell $R = \{\underline{x}: a < |\underline{x}| < b\}$ is periphRACTIC while the open sphere $R = \{\underline{x}: |\underline{x}| < b\}$ is not periphRACTIC. Also, we note that a regular region is periphRACTIC if and only if its boundary consists of more than a single closed surface.

⁴ This term was apparently introduced by Maxwell [1873](Arts.18, 22).

2.2 Theorem. There exist equilibrated stress fields on \mathcal{R} which are not totally self-equilibrated if and only if \mathcal{R} is periphRACTIC.

Proof. We show first that \mathcal{R} is periphRACTIC whenever there exists an equilibrated stress field on \mathcal{R} which is not totally self-equilibrated. Indeed, this must be so since every equilibrated stress field on \mathcal{R} is totally self-equilibrated provided \mathcal{R} is not periphRACTIC, a fact which follows from (2.1) and the divergence theorem. To prove the converse assertion assume \mathcal{R} is periphRACTIC, from which it follows that there exists an S contained in \mathcal{R} enclosing a point which does not belong to \mathcal{R} . Choose the origin of the coordinates at this point and consider the stress field

$$\tau_{km}(\underline{x}) = \frac{\lambda_1 x_1 x_k x_m}{|\underline{x}|^5}, \quad \underline{x} \in \mathcal{R} \quad (\lambda_1 \neq 0 \dots \text{const.}). \quad (2.6)$$

Clearly τ_{km} is equilibrated. Moreover, an elementary computation establishes that

$$L_k(S) = \int_S T_k dA = -\frac{4\pi}{3} \lambda_k \neq 0, \quad (2.7)$$

and consequently τ_{km} is not totally self-equilibrated.

3. The Beltrami representation.

We consider next the Beltrami stress functions and cite a well known fact, the truth of which may be confirmed by direct substitution.

3.1 Theorem. Let $\varphi_{ij} \in C^3(R)$, with $\varphi_{ij} = \varphi_{ji}$, and let

$$\tau_{km} = \varepsilon_{kip} \varepsilon_{mjQ} \varphi_{ij,pQ} \quad \text{on } R. \quad (3.1)$$

Then τ_{km} is an equilibrated stress field on R .

An equilibrated stress field τ_{km} on R which admits the representation (3.1) in terms of a symmetric tensor field $\varphi_{ij} \in C^3(R)$ is said to admit a Beltrami representation on R . With a view toward examining the completeness of this representation we state and prove the following theorem, which is due to Rieder [1960].

3.2 Theorem. Let τ_{km} admit a Beltrami representation on R .

Then τ_{km} is totally self-equilibrated.

Proof. Let S be an arbitrarily chosen closed regular surface contained in R . Then (2.2), (2.3), (3.1) imply

$$L_k(S) = \int_S \tau_{km} n_m dA = \int_S \epsilon_{kip} \epsilon_{mjq} \varphi_{ij,pq} n_m dA. \quad (3.2)$$

Fix k and let $v_j = \epsilon_{kip} \varphi_{ij,p}$, so that (3.2) becomes

$$L_k(S) = - \int_S \epsilon_{mqj} v_{j,q} n_m dA. \quad (3.3)$$

But it follows from a well-known corollary of Stokes' theorem that if $v_i \in C^1(R)$, then

$$\int_S \epsilon_{mqj} v_{j,q} n_m dA = 0 \quad (3.4)$$

and hence $L_k(S) = 0$. Similarly, in view of the identity

$$\epsilon_{ijk} \epsilon_{ipq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp} \quad (3.5)$$

and since $\varphi_{ij} = \varphi_{ji}$, one has

$$\begin{aligned}
 M_1(S) &= \int_S \varepsilon_{ijk} x_j \tau_{km} n_m dA = \int_S \varepsilon_{ijk} x_j \varepsilon_{klp} \varepsilon_{mrq} \varphi_{lr,pq} n_m dA \\
 &= \int_S \varepsilon_{mrq} (x_j \varphi_{ir,jq} - x_j \varphi_{jr,iq}) n_m dA \\
 &= \int_S \varepsilon_{mrq} (x_j \varphi_{ir,j} - \varphi_{ir} x_j - x_j \varphi_{jr,i})_{,q} n_m dA = 0
 \end{aligned}
 \quad \left. \vphantom{\int_S} \right\} (3.6)$$

Thus τ_{km} is totally self-equilibrated and the proof is complete.

The following result is an immediate consequence of Theorems 2.2 and 3.2.

3.3 Theorem. Let R be periphRACTIC. Then there exist equilibrated stress fields on R which do not admit a Beltrami representation.

In fact the equilibrated stress field defined by (2.5) on the spherical shell $R = \{\underline{x} : a < |\underline{x}| < b\}$ does not admit such a representation. Thus the Beltrami solution, and hence also the Maxwell and Morera solutions, are incomplete for periphRACTIC regions. This important limitation appears to have been largely overlooked in the literature. It is natural to ask what class of equilibrated stress fields does admit a Beltrami representation regardless of whether or not the region is periphRACTIC. The next theorem answers this question.

3.4 Theorem. (Completeness of the Beltrami representation).

Let \bar{R} be a regular region whose boundary consists of the closed surfaces B_α ($\alpha=1,2,\dots,N$) and suppose each B_α is three times continuously differentiable. Let τ_{km} have the following properties:

$$(a) \quad \tau_{km} \in C^2(\bar{R}) , \quad \tau_{km} \in C^3(R);$$

(b) τ_{km} is a totally self-equilibrated stress field on \bar{R} .

Then τ_{km} admits a Beltrami representation on R .

By virtue of Theorem 2.1, the hypothesis (b) may be replaced by the seemingly weaker assumption that τ_{km} is an equilibrated stress field on \bar{R} whose resultant force and moment vanish on each B_α . For the special case $N=1$ the region \bar{R} is not periphractic and thus, according to Theorem 2.2, the hypothesis that τ_{km} is totally self-equilibrated may be replaced by the requirement that τ_{km} is merely equilibrated.⁵

A proof of Theorem 3.4 may be based on the theorem of the vector potential. Indeed, Theorem 3.4 may be regarded as a tensorial counterpart of the latter theorem, which we now cite.

3.5 Theorem. (Existence of a vector potential.) Let \bar{R} and B_α meet the same hypotheses as in Theorem 3.4. Let v_1 have the following properties:

$$(a) \quad v_1 \in C^2(\bar{R}) , \quad v_1 \in C^3(R);$$

$$(b) \quad \int_S v_1 n_1 dA = 0 \text{ for every closed regular surface}$$

S contained in \bar{R} .

Then there exist $\omega_1 \in C^3(R)$ such that

$$v_1 = \epsilon_{1jk} \omega_{k,j} \quad \text{on } R. \quad (3.7)$$

A proof of this theorem is given by Lichtenstein [1929](pp.101-106).

⁵ The completeness proofs given by Morinaga and Nôno [1950], Ornstein [1954], Günther [1954], and Dorn and Schild [1956] are, in fact, valid only for $N=1$ since in both of these investigations the restriction that τ_{km} be totally self-equilibrated is omitted. (See Theorem 3.3.)

Proof of Theorem 3.4. The added restriction that τ_{km} is totally self-equilibrated allows us to use Günther's [1954] completeness proof in the present circumstances. In order to render the present paper sensibly self-contained we include the following version of Günther's argument.

Define B_{ijkl} on R through

$$B_{ijkl} = \varepsilon_{mik} \varepsilon_{njl} \tau_{mn} . \quad (3.8)$$

Then (3.5), (3.8), and the second of (2.1) imply

$$\tau_{ij} = \frac{1}{4} \varepsilon_{imn} \varepsilon_{jkl} B_{mknl} , \quad (3.9)$$

and

$$B_{ijkl} = B_{kl ij} = -B_{kjil} = -B_{il kj} = B_{jilk} . \quad (3.10)$$

Since τ_{km} is a totally self-equilibrated stress field

$$\int_S \tau_{ij} n_j dA = 0 , \quad \int_S (\varepsilon_{ijk} x_j \tau_{km}) n_m dA = 0 \quad (3.11)$$

for every closed regular surface S contained in R . Thus and by Theorem 3.5 there exist $b_{ij} \in C^3(R)$, $c_{ij} \in C^3(R)$ such that

$$\tau_{ij} = \varepsilon_{jpq} b_{iq,p} , \quad \varepsilon_{ijk} x_j \tau_{km} = \varepsilon_{mpq} c_{iq,p} \quad (3.12)$$

From (3.5), (3.9), (3.10), (3.12) follows

$$B_{ijkl} = \varepsilon_{pjl} (b_{pk,i} - b_{pi,k}) , \quad x_k \varepsilon_{mjl} B_{jilk} = 2 \varepsilon_{mjl} c_{il,j} . \quad (3.13)$$

Now define

$$D_{ijk} = \frac{1}{2} \varepsilon_{pjk} b_{pi} = -D_{ikj} . \quad (3.14)$$

Then the first of (3.13) becomes

$$B_{ijkl} = 2[D_{kjl,i} - D_{ijl,k}]. \quad (3.15)$$

Further, from the second of (3.13), (3.14), (3.15),

$$\varepsilon_{mjl}[c_{il,j} - 2(x_p D_{l1p})_{,j} + 2D_{l1j}] = 0. \quad (3.16)$$

Accordingly the tensor multiplying ε_{mjl} is skew symmetric with respect to j and l and hence

$$(c_{il} - 2x_p D_{l1p})_{,j} - (c_{lj} - 2x_p D_{j1p})_{,l} = 2(D_{j1l} - D_{l1j}). \quad (3.17)$$

Next define

$$F_{il} = c_{il} - 2x_p D_{l1p} \quad (3.18)$$

and conclude from (3.17) that

$$F_{il,j} - F_{lj,l} = 2(D_{j1l} - D_{l1j}). \quad (3.19)$$

Thus, and by (3.14),

$$F_{(im),k} - F_{(km),i} + F_{[ki],m} = -2D_{mik}. \quad (3.20)$$

Consequently, (3.15) implies

$$B_{ijkl} = -F_{(kj),li} - F_{(li),kj} + F_{(kl),ij} + F_{(ij),kl} \quad (3.21)$$

Finally, let $\varphi_{ij} = F_{(ij)}$. It is clear from (3.14), (3.18), and the smoothness of b_{ij} , c_{ij} that $\varphi_{ij} \in C^3(R)$. Moreover $\varphi_{ij} = \varphi_{ji}$ and (3.9), (3.21), imply (3.1). This completes the proof.

4. A generalization of the Beltrami representation.

In this section we establish a solution of (2.1) in terms of stress functions which is complete even if τ_{km} is not totally self-equilibrated and regardless of whether or not the region is periphractic.

A function $f \in C^4(R)$ that satisfies $\nabla^4 f = 0$ on R will be called biharmonic on R . The next theorem, which is readily verified by substitution, supplies a solution of (1.1) in terms of a biharmonic vector field.

4.1 Theorem. Let ψ_1 be biharmonic on R and let

$$\tau_{km} = \nabla^2(\psi_{k,m} + \psi_{m,k}) - \psi_{j,jkm} \quad \text{on } R. \quad (4.1)$$

Then τ_{km} is an equilibrated stress field on R .

The solution (4.1) is evidently incomplete since it implies that τ_{km} is biharmonic. Now consider the stress field defined by

$$\tau_{km} = \epsilon_{kip} \epsilon_{mjq} \varphi_{1j,pq} + \nabla^2(\psi_{k,m} + \psi_{m,k}) - \psi_{j,jkm} \quad \text{on } R \quad (4.2)$$

where

$$\varphi_{1j} = \varphi_{j1}, \quad \nabla^4 \psi_1 = 0 \quad \text{on } R \quad (4.3)$$

By Theorems 3.1 and 4.1 τ_{km} is an equilibrated stress field on R . The representation (4.2) of an equilibrated stress field τ_{km} in terms of a symmetric tensor field $\varphi_{1j} \in C^3(R)$ and a biharmonic vector field ψ_1 will be called a generalized Beltrami representation on R .

4.2 Theorem. (Completeness of the generalized Beltrami representation.) Let \bar{R} be an integrable region. Let τ_{km} be an equilibrated stress field on \bar{R} and suppose $\tau_{km} \in C^3(R)$. Then τ_{km} admits a generalized Beltrami representation on R .

The following trivial lemma will facilitate the proof of Theorem 4.2.

4.3 Lemma. Let \bar{R} be integrable and let $f \in C^0(\bar{R})$, $f \in C^N(R)$, with $N \geq 2$. Then the equation

$$\nabla^4 g = f \quad \text{on } R \quad (4.4)$$

has a solution $g \in C^{N+2}(R)$.

Proof. Define ρ and g on \bar{R} through

$$\rho(x) = -\frac{1}{4\pi} \int_R \frac{f(\xi)}{|x-\xi|} dV_\xi, \quad g(x) = -\frac{1}{4\pi} \int_R \frac{\rho(\xi)}{|x-\xi|} dV_\xi \quad (4.5)$$

Then⁶ $\rho \in C^{N+1}(R)$, $g \in C^{N+2}(R)$ and $\nabla^2 \rho = f$, $\nabla^2 g = \rho$ on R . Thus g is a solution of (4.4).

Proof of Theorem 4.2. By hypothesis and Lemma 4.3 there exists functions $f_{1j} \in C^5(R)$, with $f_{1j} = f_{j1}$, such that

$$\nabla^4 f_{1j} = \tau_{1j} \quad \text{on } R. \quad (4.6)$$

Further, one has the identity

$$\nabla^4 f_{1j} = \varepsilon_{1pk} \varepsilon_{jsl} \varepsilon_{pqr} \varepsilon_{stu} f_{qt,ruk} + \nabla^2 (f_{1p,pj} + f_{pj,p1}) - f_{pq,pq1j} \quad (4.7)$$

as is readily verified by expanding the first term on the right hand side with the aid of (3.5). Now define

⁶ See, for example, Courant [1962](p.246).

$$\varphi_{ps} = \varepsilon_{pqr} \varepsilon_{stu} f_{qt,ru}, \quad \psi_1 = f_{1p,p} \quad \text{on } R \quad (4.8)$$

so that $\varphi_{1j} \in C^3(R)$, with $\varphi_{1j} = \varphi_{j1}$, and $\psi_1 \in C^4(R)$. Also, (4.6), (4.7), (4.8) imply (4.2). Thus the proof is complete since $\nabla^4 \psi_1 = 0$ on R according to (4.2) and the first of (2.1).

It is of interest to note that the smoothness hypotheses which are sufficient to prove the completeness of the generalized Beltrami representation (Theorem 4.2) are much less stringent than those used to prove the completeness of the Beltrami representation (Theorem 3.4).

The next theorem is an immediate consequence of Theorems 4.2, 3.1, 3.2, 4.1.

4.4 Theorem. (Decomposition of an equilibrated stress field).

Let R be an integrable region. Let τ_{km} be an equilibrated stress field on R and suppose $\tau_{km} \in C^3(R)$. Then τ_{km} admits the decomposition

$$\tau_{km} = \tau'_{km} + \tau''_{km} \quad \text{on } R$$

where τ'_{km} is a totally self-equilibrated stress field on R , while τ''_{km} is an equilibrated stress field which is biharmonic on R .

The theorems proved in this paper are strictly analogous to certain results in vector analysis. It follows from Helmholtz's theorem that every sufficiently smooth vector field v_1 which is defined on a suitably regular region R and meets

$$v_{1,i} = 0 \quad \text{on } R, \quad (4.10)$$

admits the representation

$$v_1 = \varepsilon_{ijk} \omega_{k,j} + \mathcal{V}_{,1}, \quad \nabla^2 \mathcal{V} = 0 \quad \text{on } R. \quad (4.11)$$

Further, it is a direct consequence of (4.11) that v_1 may be written in the form

$$v_1 = v_1' + v_1'' \quad \text{on } R, \quad (4.12)$$

where the total flux of v_1' across every closed surface S in R vanishes, i.e.

$$\int_S v_1' n_1 dA = 0, \quad (4.13)$$

whereas v_1'' is harmonic. The stress equations of equilibrium are an analog of (4.10), the generalized Beltrami representation is a counterpart of (4.11), and the decomposition (4.9) is analogous to (4.12). As is apparent from Theorem 3.5, the harmonic potential \mathcal{V} in (4.11) may be set equal to zero (without loss of completeness) provided the total flux of v_1 across every closed surface in R vanishes. Analogously, we conclude from Theorem 3.4 that the biharmonic stress function ψ_k in (4.2) may be set equal to zero whenever the resultant force and moment of τ_{km} vanishes on every closed surface in R .

Consider finally the inhomogeneous stress equations of equilibrium

$$\tau_{km,m} + F_k = 0, \quad \tau_{km} = \tau_{mk} \quad \text{on } R \quad (4.14)$$

in which F_k denotes the body force density. Since we have already established a complete solution of (4.14) with $F_k = 0$,

the case in which the body forces fail to vanish is disposed of if we exhibit a particular solution of (4.14). Such a solution is given by (4.1) provided ψ_k satisfies

$$\nabla^4 \psi_k = -F_k \quad \text{on } R. \quad (4.15)$$

A solution of (4.15), in turn, is easily exhibited with the aid of an iterated Newtonian potential such as the one used in the proof of Lemma 4.3.

Acknowledgement. The author wishes to express his appreciation to E. Sternberg for suggesting this investigation and for his many helpful criticisms of the manuscript.

References

- 1863 Airy, G. B.: On the strains in the interior of beams, Phil. Trans. Roy. Soc. London, 153, 49-80.
- 1870 Maxwell, J. C.: On reciprocal figures, frames, and diagrams of force, Trans. Roy. Soc. Edinburgh, 26, 1-40.
- 1873 Maxwell, J. C.: A treatise on electricity and magnetism, Oxford.
- 1892 Morera, G.: Soluzione generale della equazioni indefinite dell'equilibrio di un corpo continuo, Atti. Accad. Nazl. Lincei, 1, 137-141.
- 1892 Beltrami, E.: Osservazioni sulla Nota precedente, Atti. Accad. Nazl. Lincei, 1, 141-142.
- 1912 Gwyther, R. F.: The formal specification of the elements of stress in cartesian and in cylindrical and spherical polar coordinates, Mem. Manchester Lit. Phil. Soc., 56, 10.
- 1929 Kellogg, O. D.: Foundations of potential theory, Berlin, Springer.
- 1929 Lichtenstein, L.: Grundlagen der Hydromechanik, Berlin, Springer.
- 1934 Finzi, B.: Integrazione delle equazioni indefinite della meccanica dei sistemi continui, 19, 6, 578-584.
- 1945 Kuzmin, R. O.: On Maxwell's and Morera's formulae in the theory of elasticity, C. R. Acad. Sci. URSS (Doklady), 49, 326-328.
- 1948 Weber, C.: Spannungsfunktionen des Dreidimensionalen Kontinuums, Z. Angew. Math. Mech., 28, 193-197.
- 1949 Krutkov, Y. A.: The tensor of stress-functions and general solutions in elastostatics (in Russian), Moscow.
- 1950 Blokh, V. I.: Stress-functions in the theory of elasticity (in Russian), Prikl. Mat. Mekh., 14, 415-422.
- 1950 Morinaga, K. and Nôno, T.: On stress-functions in general coordinates, J. Sci. Hiroshima Univ., A 14, 181-194.

- 1953 Schaefer, H.: Die Spannungsfunktionen des dreidimensionalen Kontinuums und des elastischen Körpers, Z. Angew. Math. Mech., 33, 356-362.
- 1954 Langhaar H. and Stippes M.: Three-dimensional stress functions, J. Franklin Inst., 258, 371-382.
- 1954 Ornstein, W.: Stress functions of Maxwell and Morera, Quart. Appl. Math., 2, 198-201.
- 1954 Günther, W.: Spannungsfunktionen und Vertraglichkeitsbedingungen der Kontuumsmechanik, Abh. Braunschweig Wiss. Ges., 6, 207-219.
- 1954 Kröner, E.: Die Spannungsfunktionen der dreidimensionalen isotropen Elastizitätstheorie, 139 175-188.
- 1955 Marguerre, K.: Ansätze zur Lösung der Grundgleichungen der Elastizitätstheorie, Z. Angew. Math. Mech., 35, 241-263.
- 1956 Dorn, W. S. and A. Schild: A converse of the virtual work theorem for deformable solids, Quart. Appl. Math., 14, 2, 209-213.
- 1959 Truesdell, C.: Invariant and complete stress functions for general continua, Arch. Rat. Mech. Anal., 1, 4, 1-29.
- 1960 Rieder, G.: Topologische Fragen in der Theorie der Spannungsfunktionen, Abh. Braunschweig. Wiss. Ges., 7, 4-65.
- 1960 Sternberg, E.: On some recent developments in the linear theory of elasticity, Structural mechanics, New York, Pergamon.
- 1962 Courant R.: Partial differential equations, Volume 2 of Methods of mathematical physics by R. Courant and D. Hilbert, New York, Interscience.